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Journal of Geometry and Physics 56 (2006) 1623–1636
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Twistor spaces of generalized complex structures

Johann Davidov*, Oleg Mushkarov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 8, 1113 Sofia, Bulgaria

Received 9 June 2005; received in revised form 26 August 2005; accepted 1 September 2005
Available online 4 November 2005

Abstract

The twistor construction is applied for obtaining examples of generalized complex structures (in the sense of Hitchin) that are not induced by a complex or a symplectic structure.

MSC: 53C15; 53C28

Keywords: Generalized complex structures; Twistor spaces

1. Introduction

The notion of a generalized complex structure has been introduced by Hitchin [18]. It generalizes both the concept of a complex structure and that of a symplectic one and can be considered as a complex analog of the notion of a Dirac structure introduced by Courant and Weinstein [11,12] to unify Poisson and presymplectic geometries. The generalized complex geometry has been further developed by Gualtieri [17] and has recently attracted the interest of many mathematicians and physicists, see, for example, [1,3,4,6,9,10,16,19–24,26,27,34,35] and the literature quoted therein.

A generalized complex structure on a smooth manifold M is an endomorphism J of the bundle $TM \oplus T^*M$ satisfying the following conditions: (a) $J^2 = -Id$, (b) J preserves the natural metric $< X + \xi, Y + \eta >= \frac{1}{2}(\xi(Y) + \eta(X))$, $X, Y \in TM$, $\xi, \eta \in T^*M$, (c) the $+i$-eigensubbundle of J in $(TM \oplus T^*M) \otimes \mathbb{C}$ is involutive with respect to the bracket introduced by Courant [11]. If J satisfies only the conditions (a) and (b), it is called generalized almost complex structure. The integrability condition (c) is equivalent to the vanishing of the Nijenhuis tensor of J defined by means of the Courant bracket instead of the Lie one. Every complex or symplectic structure induces a general-

* Corresponding author. Tel.: +359 2 979 38 00; fax: +359 2 971 36 49.
E-mail addresses: jtd@math.bas.bg (J. Davidov); muskarov@math.bas.bg (O. Mushkarov).

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doi:10.1016/j.geomphys.2005.09.001

ized complex structure in a natural way. Examples of generalized complex structures that cannot be obtained from a complex or a symplectic structure have been given in [8,9,17]. The main purpose of the present paper is to provide other examples of this type by means of the twistor construction.

The twistor theory has been created by Penrose [29,30] to solve problems in Mathematical Physics. The construction of a twistor space in the framework of Riemannian Geometry has been developed by Atiyah, Hitchin and Singer [2]. Following the Penrose ideas, they have defined a natural almost complex structure on the twistor space of an oriented Riemannian four-dimensional manifold and found its integrability condition. Eeells and Salamon [14] have introduced another almost complex structure on the twistor space of a four-manifold which although is never integrable plays an important role in the harmonic maps theory. The twistor construction has been generalized to any even-dimensional Riemannian manifold by Bérard-Bergery and Ochiai [5], O'Brian and Rawnsley [28], Dubois-Violette [13], Skornyakov [32]. It has been extended to the class of quaternionic Kähler manifolds by Salamon [31], Bérard-Bergery (unpublished, see [7, Sec. 14G], [5]), LeBrun [25].

Let M be an even-dimensional smooth manifold. Following the general scheme of the twistor construction, we consider the bundle $\mathcal{G}$ over M whose fibre at a point $p \in M$ consists of all generalized complex structures on the vector space T_pM (i.e. endomorphisms satisfying conditions (a) and (b) above) which yield the canonical orientation of $T_pM \oplus T_p^*M$. The general fibre of $\mathcal{G}$ admits a complex structure (in the usual sense) and this fact allows one to define two natural generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on the manifold $\mathcal{G}$ when the base manifold M is endowed with a linear connection. These are analogs of the Atiyah–Hitchin–Singer and Eells–Salamon almost complex structures. Suppose that the given connection is torsion-free. Under this condition, the main result of the present paper states that if $\dim M = 2$, then the structure $\mathcal{J}_1$ is always integrable, while if $\dim M \geq 4$, it is integrable if and only if the given connection on M is flat (i.e. M is an affine manifold). In contrast, the structure $\mathcal{J}_2$ is never integrable. The complex structure on the fibres of $\mathcal{G}$ is Kählerian with respect to a natural metric induced by the metric $<,>$. The corresponding Kähler form yields a generalized complex structure on the general fibre of $\mathcal{G}$ and one can define two new generalized almost complex structures on $\mathcal{G}$. It is not hard to see that these structures are never integrable.

One may also consider the bundle of generalized complex structures inducing the orientation opposite to the canonical one. In this case the structure $\mathcal{J}_1$ is integrable if and only if the given torsion-free connection is flat, whereas $\mathcal{J}_2$ is never integrable.

2. Generalized complex structures

Let W be a n-dimensional real vector space and g a metric of signature (p, q) on it, $p + q = n$. We shall say that an orthogonal basis $\{e_1, \dots, e_n\}$ of W is *orthonormal* if $||e_1||^2 = \cdots = ||e_p||^2 = 1$, $||e_{p+1}||^2 = \cdots = ||e_{p+q}||^2 = -1$. If $n = 2m$ is an even number and $p = q = m$, the metric g is usually called *neutral*. Recall that a complex structure J on W is called *compatible* with the metric g, if the endomorphism J is g-skew-symmetric. Suppose that $p = 2k$ and $q = 2l$, and let J be a compatible complex structure on W. Then it is easy to see by induction that there is an orthonormal basis $\{e_1, \dots, e_n\}$ of W such that $e_{2i} = Je_{2i-1}$, $i = 1, \dots, k + l$.

Now let V be a $2n$-dimensional real vector space and V^* its dual space. Then the vector space $W = V \oplus V^*$ admits a natural neutral metric defined by

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}(\xi(Y) + \eta(X)) \tag{1}$$

(we refer to [17] for algebraic facts about this metric).

Lemma 1. *Let V be a* $2n$*-dimensional real vector space and let* $\{e_i + \eta_i\}$, $i = 1, \ldots, 4n$, *be an orthonormal basis of the space* $V \oplus V^*$ *endowed with the neutral metric* (1). *Then* $e_1, \ldots, e_{2n}$ *is a basis of V and* $\eta_1, \ldots, \eta_{2n}$ *is a basis of* V^* (*similarly for* $e_{2n+1}, \ldots, e_{4n}$ *and* $\eta_{2n+1}, \ldots, \eta_{4n}$).

Proof. Assume that $\sum_{k=1}^{2n} \lambda_k e_k = 0$ for some $\lambda_1, \ldots, \lambda_{2n} \in \mathbb{R}$. Then we have $\sum_{k=1}^{2n} \lambda_k \eta_l(e_k) = 0$ for every $l = 1, \ldots, 2n$. Denote by P the matrix $[\eta_l(e_k)]$, $1 \leq l, k \leq 2n$. Let I be the unit $2n \times 2n$-matrix. Then the matrix $S = P - I$ is skew-symmetric since $\eta_l(e_k) + \eta_k(e_l) = 2\delta_{lk}$. Therefore $(\det P)^2 = \det(P^t P) = \det(I - S^2)$. Let $\beta_1, \ldots, \beta_{2n}$ be the eigenvalues of the symmetric matrix S^2. Then $\beta_k \leq 0$ since the matrix S is skew-symmetric. Thus $(\det P)^2 = \Pi_{k=1}^{2n}(1 - \beta_k) \geq 1$, in particular $\det P \neq 0$. Therefore all λ_k' s must be zero. □

Let $\{e_i\}$ be an arbitrary basis of a real vector space V and $\{\alpha_i\}$ its dual basis, $i = 1, \ldots, 2n$. Then the orientation of the space $V \oplus V^*$ determined by the basis $\{e_i, \alpha_i\}$ does not depend on the choice of the basis $\{e_i\}$. Further on, we shall always consider $V \oplus V^*$ with this *canonical orientation*.

The next lemma is of technical character and will be used in the last section.

Lemma 2. *Let V be a* 2*-dimensional real vector space and let* $\{Q_i = e_i + \eta_i\}$, $1 \leq i \leq 4$, *be an orthonormal basis of the space* $V \oplus V^*$ *endowed with its natural neutral metric* (1). *Then*

$$e_3 = a_{11}e_1 + a_{12}e_2, \qquad e_4 = a_{21}e_1 + a_{22}e_2,$$

where $A = [a_{kl}]$ *is an orthogonal matrix. If* $\det A = 1$, *the basis* $\{Q_i\}$ *yields the canonical orientation of* $V \oplus V^*$ *and if* $\det A = -1$ *it yields the opposite one*.

Proof. According to Lemma 1,

$$e_3 = a_{11}e_1 + a_{12}e_2, \quad e_4 = a_{21}e_1 + a_{22}e_2$$

and

$$e_1 = b_{11}e_3 + b_{12}\, e_4, \quad e_2 = b_{21}e_3 + b_{22}e_4$$

where $[b_{kl}] = [a_{kl}]^{-1}$. Since the basis $\{Q_i\}$ is orthonormal, we have $\eta_i(e_j) + \eta_j(e_i) = 0, i \neq j, 1 \leq i, j \leq 4$ and $\eta_1(e_1) = \eta_2(e_2) = 1$, $\eta_3(e_3) = \eta_4(e_4) = -1$. The latter identities and the identities $\eta_1(e_3) + \eta_3(e_1) = 0$, $\eta_1(e_4) + \eta_4(e_1) = 0$, $\eta_3(e_4) + \eta_4(e_3) = 0$ imply

$$a_{12}\eta_1(e_2) + b_{12}\eta_3(e_4) = b_{11} - a_{11}, \qquad a_{22}\eta_1(e_2) - b_{11}\eta_3(e_4) = b_{12} - a_{21}.$$

It follows that

$$b_{11}^2 + b_{12}^2 = 1. \tag{2}$$

Similarly, we see that

$$b_{21}^2 + b_{22}^2 = 1, \quad a_{11}^2 + a_{12}^2 = 1, \quad a_{21}^2 + a_{22}^2 = 1. \tag{3}$$

Expressing b_{kl}' s in terms of a_{kl}' s, we get from (2) and (3) that

$$(a_{11}a_{22} - a_{12}a_{21})^2 = 1.$$

It follows that the matrix $A = [a_{kl}]$ is orthogonal.

To prove the second part of the lemma, let us denote by $\{\alpha_1, \alpha_2\}$ the dual basis of the basis $\{e_1, e_2\}$ of V. Then

$$\eta_1 = \alpha_1 + c\alpha_2, \qquad \eta_2 = -c\alpha_1 + \alpha_2, \quad \eta_3 = d_{11}\alpha_1 + d_{12}\alpha_2, \quad \eta_4 = d_{21}\alpha_1 + d_{22}\alpha_2$$

for some constants c and d_{kl}. For the coefficients d_{kl} we have

$$d_{11} = \eta_3(e_1) = -\eta_1(e_3) = -(a_{11} + ca_{12}),$$
$$d_{12} = \eta_3(e_2) = -\eta_2(e_3) = -(-ca_{11} + a_{12}),$$
$$d_{21} = \eta_4(e_1) = -\eta_1(e_4) = -(a_{21} + ca_{22}),$$
$$d_{22} = \eta_4(e_2) = -\eta_2(e_4) = -(-ca_{21} + a_{22}).$$

Thus, if we set $c_{11} = c_{22} = 1$ and $c_{12} = -c_{21} = c$, then $[d_{kl}] = -[a_{kl}][c_{lk}]$. Set $C = [c_{kl}]$ and let I be the unit 2×2-matrix. Then the transition matrix from the basis $\{e_1, e_2, \alpha_1, \alpha_2\}$ to the basis $\{Q_1, Q_2, Q_3, Q_4\}$ has the form

$$\begin{bmatrix} I & C \\ A & -AC^t \end{bmatrix}$$

The determinant of this matrix is equal to 4det A, which proves the lemma. □

A *generalized complex structure* on a real vector space V is, by definition, a complex structure on the space $V \oplus V^*$ compatible with its natural neutral metric [18]. If a vector space admits a generalized complex structure, it is necessarily of even dimension [17]. We refer to [17] for more facts about the generalized complex structures. Here are some basic examples of such structures [17–19].

Example 1. Let K be a complex structure on V and define a complex structure K^* on V^* by setting $(K^*\alpha)(X) = \alpha(KX)$, $\alpha \in V^*$, $X \in V$. Then the endomorphism J on $V \oplus V^*$ defined by $J = K$ on V and $J = -K^*$ on V^* is a generalized complex structure on V. This structure yields the canonical orientation of $V \oplus V^*$ (the orientation induced by J is defined by means of a basis of the form $Q_1, JQ_1, \ldots, Q_{2n}, JQ_{2n}$).

Example 2. Let ω be a symplectic form on V (i.e. a non-degenerate 2-form). Then the map $X \to \iota_X\omega$ is an isomorphism of V onto V^*. Denote this isomorphism also by ω and define a complex structure S on $V \oplus V^*$ by setting $SX = \omega(X)$ and $S\alpha = -\omega^{-1}(\alpha)$ for $X \in V$ and $\alpha \in V^*$. Then S is compatible with the natural neutral metric of $V \oplus V^*$, so S is a generalized complex structure on V. The structure S induces the canonical orientation of $V \oplus V^*$ if and only if $n = \frac{1}{2}\dim V$ is an even number.

Now let g be a metric on V (of any signature) and K a complex structure on V compatible with the metric g. Then the generalized complex structures J and S yielded by K and the 2-form $\omega(X, Y) = g(KX, Y)$ commute.

Example 3. The direct sum of generalized complex structures is also a generalized complex structure.

Example 4. Any 2-form $B \in \Lambda^2 V^*$ acts on $V \oplus V^*$ via the inclusion $\Lambda^2 V^* \subset \Lambda^2(V \oplus V^*) \cong so(V \oplus V^*)$; in fact this is the action $X + \xi \to \iota_X B$; $X \in V$, $\xi \in V^*$. Denote the latter map again by B. Then the invertible map e^B is given by $X + \xi \to X + \xi + \iota_X B$ and is an orthogonal transformation of $V \oplus V^*$. Thus, given a generalized complex structure J on V, the map $e^B J e^{-B}$ is also a generalized complex structure on V, called the B-transform of J.

Similarly, any two-vector $\beta \in \Lambda^2 V$ acts on $V \oplus V^*$. If we identify V with $(V^*)^*$, so $\Lambda^2 V \cong \Lambda^2(V^*)^*$, the action is given by $X + \xi \to \iota_\xi \beta \in V$. Denote this map by β. Then the exponential map e^β acts on $V \oplus V^*$ via $X + \xi \to X + \iota_\xi\beta + \xi$, in particular e^β is an orthogonal transformation. Hence, if J is a generalized complex structure on V, so is $e^\beta J e^{-\beta}$. It is called the β-transform of J.

Let W be a $2m$-dimensional real vector space equipped with a metric g of signature $(2p, 2q)$, $p+q=m$. Denote by $J(W)$ the set of all complex structures on W compatible with the metric g. The group $O(g)$ of orthogonal transformations of W acts transitively on $J(W)$ by conjugation and $J(W)$ can be identified with the homogeneous space $O(2p, 2q)/U(p, q)$. In particular, $\dim J(W) = m^2 - m$. The group $O(2p, 2q)$ has four connected components, while $U(p, q)$ is connected, therefore $J(W)$ has four components.

Example 5. The space $O(2, 2)/U(1, 1)$ is the disjoint union of two copies of the hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$. It seems instructive to see this in the context of compatible complex structures. Let W be a 4-dimensional real vector space equipped with a neutral metric and $e_1, \ldots, e_4$ an orthonormal basis of W. Set $\varepsilon_k = ||e_k||^2$, $k = 1, \ldots, 4$, and define skew-symmetric endomorphisms of W by setting $S_{ij}e_k = \varepsilon_k(\delta_{ik}e_j - \delta_{kj}e_i)$, $1 \le i, j, k \le 4$. Then the endomorphisms

$$I_1 = S_{12} - S_{34}, \qquad I_2 = S_{13} - S_{24}, \qquad I_3 = S_{14} + S_{23},$$

$$J_1 = S_{12} + S_{34}, \qquad J_2 = S_{13} + S_{24}, \qquad J_3 = S_{14} - S_{23},$$

constitute a basis of the space of skew-symmetric endomorphisms of W subject to the following relations: $I_1^2 = -Id$, $I_2^2 = I_3^2 = Id$, $J_1^2 = -Id$, $J_2^2 = J_3^2 = Id$, $I_rI_s = -I_sI_r$, $J_rJ_s = -J_sJ_r$, $1 \le r \ne s \le 3$ and $I_rJ_s = J_sI_r$, $1 \le r, s \le 3$. Let K be a complex structure on W compatible with the metric and let us set $K = \sum_{r=1}^{3}(x_rI_r + y_rJ_r)$. Then we have

$$(-x_1^2 + x_2^2 + x_3^2 - y_1^2 + y_2^2 + y_3^2)Id + 2\sum_{r,s=1}^{3} x_ry_sI_rJ_s = -Id$$

Evaluating the latter identity at $e_1, \ldots, e_4$, we see that

$$-x_1^2 + x_2^2 + x_3^2 - y_1^2 + y_2^2 + y_3^2 = -1, \quad \text{and} \quad x_ry_s = 0 \quad \text{for} \quad r, s = 1, 2, 3.$$

Therefore $K^2 = -Id$ if and only if either $x_1^2 - x_2^2 - x_3^2 = 1$ and $y_1 = y_2 = y_3 = 0$ or $y_1^2 - y_2^2 - y_3^2 = 1$ and $x_1 = x_2 = x_3 = 0$.

Consider $J(W)$ as a (closed) submanifold of the vector space $so(g)$ of g – skew-symmetric endomorphisms of W. Then the tangent space of $J(W)$ at a point J consists of all endomorphisms $Q \in so(g)$ anti-commuting with J. Thus we have a natural $O(g)$ – invariant almost complex structure $\mathcal{K}$ on $J(W)$ defined by $\mathcal{K}Q = J \circ Q$. It is easy to check that this structure is integrable.

Fix an orientation on W and denote by $J^{\pm}(W)$ the set of compatible complex structures on W that induce $\pm$ the orientation of W. The set $J^{\pm}(W)$ has the homogeneous representation $SO(2p, 2q)/U(p, q)$ and, thus, is the union of two components of $J(W)$.

Example 6. Under the notations of Example 5, let $e_1, \ldots, e_4$ be an oriented orthonormal basis of W. Then it is easy to see that $J^{+}(W)$ is the hyperboloid $\{\sum_{r=1}^{3} x_rI_r : x_1^2 - x_2^2 - x_3^2 = 1\}$.

Further on, given an even-dimensional real vector space V, the set $J^{+}(V \oplus V^*)$ of generalized complex structures on V inducing the canonical orientation of $V \oplus V^*$ will be denoted by $G(V)$.

The group $GL(V)$ acts on $V \oplus V^*$ by letting $GL(V)$ act on V^* in the standard way. This action preserves the neutral metric (1) and the canonical orientation of $V \oplus V^*$. Thus, we have an embedding of $GL(V)$ into the group $SO(<, >)$ and, via this embedding, $GL(V)$ acts on the manifold $G(V)$ in a natural manner.

A generalized almost complex structure on an even-dimensional smooth manifold M is, by definition, an endomorphism J of the bundle $TM \oplus T^*M$ with $J^2 = -Id$ which preserves the natural neutral metric of $TM \oplus T^*M$. Such a structure is said to be *integrable* or *a generalized complex structure* if its $+i$-egensubbunle of $(TM \oplus T^*M) \oplus \mathbb{C}$ is closed under the Courant bracket

[18]. Recall that if X, Y are vector fields on M and ξ, η are 1-forms, the Courant bracket [11] is defined by the formula:

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(\imath_X\eta - \imath_Y\xi),$$

where $[X, Y]$ on the right hand-side is the Lie bracket and $\mathcal{L}$ means the Lie derivative. As in the case of almost complex structures, the integrability condition for a generalized almost complex structure J is equivalent to the vanishing of its Nijenhuis tensor N, the latter being defined by means of the Courant bracket:

$$N(A, B) = -[A, B] - J[A, JB] - J[JA, B] + [JA, JB], \quad A, B \in TM \oplus T^*M.$$

Example 7. Let J be a generalized almost complex structure on a manifold M and let B be a smooth 2-form on M. Then, according to Example 4, $e^B Je^{-B}$ is a generalized almost complex structure on M. The exponential map e^B is an authomorphism of the Courant bracket (i.e. $[e^B(X + \xi), e^B(Y + \eta)] = e^B[X + \xi, Y + \eta]$) if and only if the form B is closed [17]. In this case the structure $e^B Je^{-B}$ is integrable exactly when the structure J is so.

3. The twistor space of generalized complex structures

Let M be a smooth manifold of dimension $2n$. Denote by $\pi : \mathcal{G} \to M$ the bundle over M whose fibre at a point $p \in M$ consists of all generalized complex structures on T_pM that induce the canonical orientation of $T_pM \oplus T_p^*M$. This is the associated bundle

$$GL(M) \times_{GL(2n,\mathbb{R})} G(\mathbb{R}^{2n}),$$

where $GL(M)$ denotes the principal bundle of linear frames on M.

Let ∇ be a linear connection on M. Following the standard twistor construction, we can define two generalized almost complex structures $\mathcal{J}_1^\nabla$ and $\mathcal{J}_2^\nabla$ on the manifold $\mathcal{G}$ in the following way. The connection ∇ gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of the associated bundle $\mathcal{G}$ into vertical and horizontal parts. The vertical space $\mathcal{V}_J$ of $\mathcal{G}$ at a point $J \in \mathcal{G}$ is the tangent space at J of the fibre through this point. This fibre is the manifold $G(T_pM)$, $p = \pi(J)$, which admits a natural complex structure $\mathcal{K}$ defined in the previous section and we set $\mathcal{J}_\alpha^\nabla = (-1)^{\alpha+1}\mathcal{K}$ on $\mathcal{V}_J$ and $\mathcal{J}_\alpha^\nabla = (-1)^\alpha\mathcal{K}^*$ on $\mathcal{V}_J^*$, $\alpha = 1, 2$. Thus $\mathcal{J}_\alpha^\nabla U = (-1)^{\alpha+1} J \circ U$ for every $U \in \mathcal{V}_J$ (U being considered as an endomorphism of $T_pM \oplus T_p^*M$) and $(\mathcal{J}_\alpha^\nabla \omega)(U) = (-1)^\alpha\omega(J \circ U)$ for $\omega \in \mathcal{V}_J^*$. The horizontal space $\mathcal{H}_J$ is isomorphic via the differential π_{*J} to the tangent space T_pM, $p = \pi(J)$. Denoting the restriction of π_{*J} to $\mathcal{H}$ by $\pi_\mathcal{H}$, we define $\mathcal{J}_\alpha^\nabla$ on $\mathcal{H}_J \oplus \mathcal{H}_J^*$ as the lift of the endomorphism J by the map $\pi_\mathcal{H} \oplus (\pi_\mathcal{H}^{-1})^*$.

Note also that the analogs of $\mathcal{J}_1^\nabla$ and $\mathcal{J}_2^\nabla$ on the bundle of generalized complex structures yielding the orientation opposite to the canonical one are not induced by almost complex or symplectic structures for any n.

Remark. The fibre of $\mathcal{G}$ at any point $p \in M$ contains generalized complex structures on T_pM which do not preserve T_pM as well as structures which do not send T_pM onto T_p^*M. This shows that the generalized almost complex structures $\mathcal{J}_1^\nabla$ and $\mathcal{J}_2^\nabla$ are not induced by almost complex or symplectic structures. They are not B- or β-transforms of such structures since any fibre of $\mathcal{G}$ contains also generalized complex structures which do not preserve T_p^*M, structures preserving T_p^*M and ones preserving T_pM.

Further the generalized almost complex structure $\mathcal{J}_\alpha^\nabla$ will be simply denoted by $\mathcal{J}_\alpha$ when the connection ∇ is understood. The image of every $A \in T_pM \oplus T_p^*M$ under the map $\pi_\mathcal{H}^{-1} \oplus \pi_\mathcal{H}^*$ will be denoted by A^h. The elements of $\mathcal{H}^*_J$, resp. $\mathcal{V}^*_J$, will be considered as 1-forms on $T_J\mathcal{G}$ vanishing on $\mathcal{V}_J$, resp. $\mathcal{H}_J$.

Let $A(M)$ be the bundle of the endomorphisms of $TM \oplus T^*M$ which are skew-symmetric with respect to its natural neutral metric $\langle , \rangle$; the fibre of $A(M)$ at a point $p \in M$ will be denoted by $A_p(M)$. Consider the twistor space $\mathcal{G}$ as a subbundle of $A(M)$. Then the inclusion of $\mathcal{G}$ is fibre-preserving and the horizontal space of $\mathcal{G}$ at a point J coincides with the horizontal space of $A(M)$ at that point since the inclusion $G(\mathbb{R}^{2n}) \subset so(2n, 2n)$ is $SO(2n, 2n)$-equivariant. Let $(U, x_1, \ldots, x_{2n})$ be a local coordinate system of M and $\{Q_1, \ldots, Q_{4n}\}$ an orthonormal frame of $TM \oplus T^*M$ on U. Set $\varepsilon_k = ||Q_k||^2$, $k = 1, \ldots, 4n$, and define sections S_{ij}, $1 \leq i, j \leq 4n$, of $A(M)$ by the formula:

$$S_{ij}Q_k = \varepsilon_k(\delta_{ik}Q_j - \delta_{kj}Q_i). \tag{4}$$

Then S_{ij}, $i < j$, form an orthogonal frame of $A(M)$ with respect to the metric $\langle a, b\rangle = -\frac{1}{2}\mathrm{Trace}\,(a \circ b)$; $a, b \in A(M)$; moreover $||S_{ij}||^2 = \varepsilon_i\varepsilon_j$ for $i \neq j$. Set

$$\tilde{x}_l(a) = x_l \circ \pi(a), \qquad y_{ij}(a) = \varepsilon_i\varepsilon_j\langle a, S_{ij}\rangle$$

for $a \in A(M)$. Then $(\tilde{x}_l, y_{ij})$, $1 \leq l \leq 2n$, $1 \leq i < j \leq 4n$, is a local coordinate system of the manifold $A(M)$.

Let

$$V = \sum_{i<j} v_{ij}\frac{\partial}{\partial y_{ij}}(J)$$

be a vertical vector of $\mathcal{G}$ at a point J. It is convenient to set $v_{ij} = -v_{ji}$ for $i \geq j$, $1 \leq i, j \leq 4n$. Then the endomorphism V of $T_pM \oplus T^*_pM$, $p = \pi(J)$, is determined by $VQ_i = \sum_{j=1}^{4n}\varepsilon_i v_{ij}Q_j$. Moreover

$$\mathcal{J}_\alpha V = (-1)^{\alpha+1}\sum_{i<j}\left(\sum_{k=1}^{4n} v_{ik}y_{kj}\varepsilon_k\right)\frac{\partial}{\partial y_{ij}}. \tag{5}$$

Note also that, for every $A \in T_pM \oplus T^*_pM$, we have

$$A^h = \sum_{i=1}^{4n}(\langle A, Q_i\rangle \circ \pi)\varepsilon_i Q_i^h \quad \text{and} \quad \mathcal{J}_\alpha A^h = \sum_{i,j=1}^{4n}(\langle A, Q_i\rangle \circ \pi)y_{ij}Q_j^h. \tag{6}$$

For each vector field

$$X = \sum_{i=1}^{2n} X^i\frac{\partial}{\partial x_i}$$

on U, the horizontal lift X^h on $\pi^{-1}(U)$ is given by

$$X^h = \sum_{k=1}^{2n}(X^l \circ \pi)\frac{\partial}{\partial \tilde{x}_l} - \sum_{i<j}\sum_{p<q} y_{pq}(\langle \nabla_X S_{pq}, S_{ij}\rangle \circ \pi)\varepsilon_i\varepsilon_j\frac{\partial}{\partial y_{ij}}, \tag{7}$$

where ∇ is the connection on $A(M)$ induced by the given connection on M.

Let $a \in A(M)$ and $p = \pi(a)$. Then (7) implies that, under the standard identification of $T_aA_p(M)$ with the vector space $A_p(M)$ of skew-symmetric endomorphisms of $T_pM \oplus T^*_pM$, we have

$$[X^h, Y^h]_a = [X, Y]^h_a + R(X, Y)a, \tag{8}$$

where $R(X, Y)a$ is the curvature of the connection ∇ on $A(M)$ (for the curvature tensor we adopt the following definition: $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$).

Notations. Let $J \in \mathcal{G}$ and $p = \pi(J)$. Take an oriented orthonormal basis $\{a_1, \ldots a_{4n}\}$ of $T_pM \oplus T^*_pM$ such that $a_{2l} = Ja_{2l-1}, l = 1, \ldots, 2n$. Let $\{Q_i\}, i = 1, \ldots, 4n$, be an oriented orthonormal frame of $TM \oplus T^*M$ near the point P such that

$$Q_i(p) = a_i, \quad \text{and} \quad \nabla Q_i|_p = 0, \quad i = 1, \ldots, 4n.$$

Define a section s of $\mathcal{G}$ by setting

$$sQ_{2l-1} = Q_{2l}, \qquad sQ_{2l} = -Q_{2l-1}, \quad l = 1, \ldots, 2n.$$

Then, considering s as a section of $A(M)$, we have

$$s(p) = J, \quad \nabla s|_p = 0.$$

Thus $X^h_J = s_*X$ for every $X \in T_pM$.

Further, given a smooth manifold M, the natural projections of $TM \oplus T^*M$ onto TM and T^*M will be denoted by π_1 and π_2, respectively.

We shall use the above notations throughout this section.

To compute the Nijenhuis tensor of the generalize almost complex structure $\mathcal{J}_\alpha$, $\alpha = 1,2$, on the twistor space $\mathcal{G}$ we need some preliminary lemmas.

Lemma 3. *If A and B are sections of the bundle $TM \oplus T^*M$ near p, then*

(i) $[\pi_1(A^h), \pi_1(\mathcal{J}_\alpha B^h)]_J = [\pi_1(A), \pi_1(sB)]^h_J + R(\pi_1(A), \pi_1(JB))J.$
(ii) $[\pi_1(\mathcal{J}_\alpha A^h), \pi_1(\mathcal{J}_\alpha B^h)]_J = [\pi_1(sA), \pi_1(sB)]^h_J + R(\pi_1(JA), \pi_1(JB))J.$

Proof. Set $X = \pi_1(A)$. By (7), we have $X^h_J = \sum_{l=1}^{2n} X^l(p)\frac{\partial}{\partial \tilde{x}_l}(J)$ since $\nabla S_{ij}|_p = 0$, $i, j = 1, \ldots, 4n$. Then, using (6), we get

$$[X^h, \pi_1(\mathcal{J}_\alpha B^h)]_J = \sum_{i,j=1}^{4n} \langle B, Q_i\rangle_p y_{ij}(J)[X^h, \pi_1(Q_j)^h]_J + X_p(\langle B, Q_i\rangle)y_{ij}(J)(\pi_1(Q_j))^h_J.$$

Moreover, $sB = \sum_{ij}\langle B, Q_i\rangle(y_{ij} \circ s)Q_j$ since $(\mathcal{J}_\alpha B^h) \circ s = (sB)^h \circ s$. Now formula ($i$) follows by means of (8). Similar computations give (ii). □

Lemma 4. *Let A and B be sections of the bundle $TM \oplus T^*M$ near p, and let $Z \in T_pM$, $W \in \mathcal{V}_J$. Then*

(i) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(B^h))_J = (\mathcal{L}_{\pi_1(A)}\pi_2(B))^h_J.$
(ii) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{J}_\alpha B^h))_J = (\mathcal{L}_{\pi_1(A)}\pi_2(sB))^h_J.$
(iii) $(\mathcal{L}_{\pi_1(\mathcal{J}_\alpha A^h)}\pi_2(B^h))_J(Z^h + W) = (\mathcal{L}_{\pi_1(sA)}\pi_2(B))^h_J(Z^h) + (\pi_2(B))_p(\pi_1(WA)).$
(iv) $(\mathcal{L}_{\pi_1(\mathcal{J}_\alpha A^h)}\pi_2(\mathcal{J}_\alpha B^h))_J(Z^h + W) = (\mathcal{L}_{\pi_1(sA)}\pi_2(sB))^h_J(Z^h) + (\pi_2(JB))_p(\pi_1(WA)).$

Proof. Formula (i) follows from (8). Simple computations involving (6) and (8) give (ii)–(iv). □

The next lemma is also easy to prove by means of (6) and (8).

Lemma 5. *Let A and B be sections of the bundle $TM \oplus T^*M$ near p. Let $Z \in T_pM$ and $W \in \mathcal{V}_J$. Then*

(i) $(d\iota_{\pi_1(A^h)}\pi_2(B^h))_J = (d\iota_{\pi_1(A)}\pi_2(B))^h_J$

(ii) $(d\iota_{\pi_1(A^h)}\pi_2(\mathcal{J}_\alpha B^h))_J(Z^h + W) = (d\iota_{\pi_1(A)}\pi_2(sB))^h_J(Z^h) + (\pi_2(WB))_p(\pi_1(A))$

(iii) $(d\iota_{\pi_1(\mathcal{J}_\alpha A^h)}\pi_2(B^h))_J(Z^h + W) = (d\iota_{\pi_1(sA)}\pi_2(B))^h_J(Z^h) + (\pi_2(B))_p(\pi_1(WA))$

(iv) $(d\iota_{\pi_1(\mathcal{J}_\alpha A^h)}\pi_2(\mathcal{J}_\alpha B^h))_J(Z^h + W) = (d\iota_{\pi_1(sA)}\pi_2(sB))^h_J(Z^h) + (\pi_2(WB))_p(\pi_1(JA)) + (\pi_2(JB))_p(\pi_1(WA))$

For any (local) section a of $A(M)$, following [15], denote by $\tilde{a}$ the vertical vector field on $\mathcal{G}$ defined by

$$\tilde{a}_J = a + J \circ a \circ J. \tag{9}$$

Let us note that for every $J \in \mathcal{G}$ we can find sections $a_1, \ldots, a_m$, $m = 4n^2 - 2n$, of $A(M)$ near the point $p = \pi(J)$ such that $\tilde{a}_1, \ldots, \tilde{a}_m$ form a basis of the vertical vector space at each point in a neighbourhood of J.

Lemma 6. *Let $J \in \mathcal{G}$ and let a be a section of $A(M)$ near the point $p = \pi(J)$. Then, for any section A of the bundle $TM \oplus T^*M$ near p, we have*

(i) $[\pi_1(A^h), \tilde{a}]_J = (\widetilde{\nabla_{\pi_1(A)}a})_J$

(ii) $[\pi_1(A^h), \mathcal{J}_\alpha\tilde{a}]_J = (-1)^{\alpha+1} J \circ (\widetilde{\nabla_{\pi_1(A)}a})_J$

(iii) $[\pi_1(\mathcal{J}_\alpha A^h), \tilde{a}]_J = (\widetilde{\nabla_{\pi_1(JA)}a})_J - \pi_1(\tilde{a}(A))^h_J$

(iv) $[\pi_1(\mathcal{J}_\alpha A^h), \mathcal{J}_\alpha\tilde{a}]_J = (-1)^{\alpha+1}[J \circ (\widetilde{\nabla_{\pi_1(JA)}a})_J - \pi_1((J \circ \tilde{a})(A))^h_J]$

Proof. Let $a(Q_i) = \sum_{j=1}^{4n} \varepsilon_i a_{ij} Q_j$, $i = 1, \ldots, 4n$. Then, in the local coordinates of $A(M)$ introduced above,

$$\tilde{a} = \sum_{i<j} \tilde{a}_{ij} \frac{\partial}{\partial y_{ij}}$$

where

$$\tilde{a}_{ij} = a_{ij} \circ \pi + \sum_{k,l=1}^{4n} y_{ik}(a_{kl} \circ \pi) y_{lj} \varepsilon_k \varepsilon_l.$$

Let us also note that for every vector field X on M near the point p, in view of (7), we have

$$\left[X^h, \frac{\partial}{\partial y_{ij}}\right]_J = 0, \quad X^h_J = \sum_{i=1}^{2n} X^i(p)\frac{\partial}{\partial \tilde{x}_i}(J), \quad (\nabla_{X_p}a)(Q_i) = \sum_{j=1}^{4n} \varepsilon_i X_p(a_{ij}) Q_j$$

since $\nabla Q_i|_p = 0$ and $\nabla S_{ij}|_p = 0$. Now the lemma follows by simple computations making use of (5) and (6). □

The proof of the next lemma is easy and will be omitted.

Lemma 7. *Let A be a section of the bundle $TM \oplus T^*M$ and V a vertical vector field on $\mathcal{G}$. Then*

(i) $\mathcal{L}_V \pi_2(A^h) = 0$; $\imath_V \pi_2(A^h) = 0$.
(ii) $\mathcal{L}_V \pi_2(\mathcal{J}_\alpha A^h) = \pi_2(VA)^h$; $\imath_V \pi_2(\mathcal{J}_\alpha A^h) = 0$.

Notations. Take a point $J \in \mathcal{G}$ and fix a basis $\{U_{2r-1}, U_{2r} = \mathcal{J}_1 U_{2r-1}\}$, $r = 1, \ldots, 2n^2 - n$, of the vertical space $\mathcal{V}_J$. Now let us take sections a_{2r-1} of $A(M)$ near the point $p = \pi(J)$ such that $a_{2r-1} = U_{2r-1}$ and $\nabla a_{2r-1}|_p = 0$. Define vertical vector fields $\tilde{a}_{2r-1}$ by (9). Then $\{\tilde{a}_{2r-1}, \mathcal{J}_1 \tilde{a}_{2r-1}\}$, $r = 1, \ldots, 2n^2 - n$, is a frame of the vertical bundle on $\mathcal{G}$ near the point J. Denote by $\{\beta_{2r-1}, \beta_{2r}\}$ the dual frame. Then $\beta_{2r} = \mathcal{J}_1 \beta_{2r-1}$.

Under these notations, identity (8) and Lemmas 3 and 6 imply the following

Lemma 8. *Let A be a section of the bundle $TM \oplus T^*M$ near the point $p = \pi(J)$. Then for every $Z \in T_pM$* and $s, q = 1, \ldots, 4n^2 - 2n$, *we have*

(i) $(\mathcal{L}_{\pi_1(A^h)} \beta_s)_J(Z^h + U_q) = -\beta_s(R(\pi_1(A), Z)J)$.
(ii) $(\mathcal{L}_{\pi_1(\mathcal{J}_\alpha A^h)} \beta_s)_J(Z^h + U_q) = -\beta_s(R(\pi_1(JA), Z)J)$.

Proposition 1. *Suppose that the connection ∇ is torsion-free and let $J \in \mathcal{G}$, $A, B \in T_{\pi(J)}M \oplus T^*_{\pi(J)}M$, $V, W \in \mathcal{V}_J$, $\varphi, \psi \in \mathcal{V}^*_J$. Then*

(i) $N_\alpha(A^h, B^h)_J = -R(\pi_1(A), \pi_1(B))J + (-1)^\alpha J \circ R(\pi_1(A), \pi_1(JB))J +$
$(-1)^\alpha J \circ R(\pi_1(JA), \pi_1(B))J + R(\pi_1(JA), \pi_1(JB))J - \frac{1}{2}[1 + (-1)^\alpha]\omega_{A,B}$,
where $\omega_{A,B}$ is the vertical 1-form on $\mathcal{G}$ given by

$$\begin{aligned}\omega_{A,B}(W) &= \pi_2(JA)(\pi_1(WB)) + \pi_2(WB)(\pi_1(JA)) \\ &\quad - \pi_2(JB)(\pi_1(WA)) - \pi_2(WA)(\pi_1(JB))\end{aligned}$$

for every $W \in \mathcal{V}_J$.
(ii) $N_\alpha(A^h, V)_J = [1 + (-1)^\alpha]((J \circ V)A)^h_J$
(iii) $N_\alpha(A^h, \varphi)_J \in \mathcal{H}_J \oplus \mathcal{H}^*_J$ *and* $\langle \pi_* N_\alpha(A^h, \varphi)_J, B\rangle = -\frac{1}{2}\varphi(N_1(A^h, B^h)_J)$
$-\frac{1}{2}[1 + (-1)^\alpha]\varphi(J \circ R(\pi_1(A), \pi_1(JB))J + J \circ R(\pi_1(JA), \pi_1(B))J)$.
(iv) $N_\alpha(V + \varphi, W + \psi)_J = 0$.

Proof. Set $p = \pi(J)$ and extend the vectors A, B to (local) sections of $TM \oplus T^*M$, denoted again by A, B, in such a way that $\nabla A|_p = \nabla B|_p = 0$.

Let s be the section of $\mathcal{G}$ defined above with the property that $s(p) = J$ and $\nabla s|_p = 0$ (s being considered as a section of $A(M)$).

According to Lemmas 3–5, the part of $N_\alpha(A^h, B^h)_J$ lying in $\mathcal{H}_J \oplus \mathcal{H}^*_J$ is given by

$$(\mathcal{H} \oplus \mathcal{H}^*)N_\alpha(A^h, B^h)_J = (-[A, B] - s[A, sB] - s[sA, B] + [sA, sB])^h_J.$$

Note that we have $\nabla\pi_1(A)|_p = \pi_1(\nabla A|_p) = 0$ and $\nabla\pi_1(sA)|_p = \pi_1((\nabla s)|_p(A) + s(\nabla A|_p)) = 0$. Similarly, $\nabla\pi_2(A)|_p = 0$ and $\nabla\pi_2(sA)|_p = 0$. We also have $\nabla\pi_1(B)|_p = 0$, $\nabla\pi_1(sB)|_p = 0$ and $\nabla\pi_2(B)|_p = 0$, $\nabla\pi_2(sB)|_p = 0$. Now, since ∇ is torsion-free, we easily get $(\mathcal{H} \oplus \mathcal{H}^*)N_\alpha(A^h, B^h)_J = 0$ by means of the following simple observation. Let Z be a vector field and ω a 1-form on M such that $\nabla Z|_p = 0$ and $\nabla\omega|_p = 0$. Then for every $T \in T_pM$

$$(\mathcal{L}_Z\omega)(T)_p = (\nabla_Z\omega)(T)_p = 0, \quad \text{and} \quad (d\imath_Z\omega)(T)_p = (\nabla_T\omega)(Z)_p = 0.$$

By Lemmas 3–5 the vertical part of $N_\alpha(A^h, B^h)_J$ is equal to

$$\mathcal{V}N_\alpha(A^h, B^h)_J = -R(\pi_1(A), \pi_1(B))J - \mathcal{J}_\alpha R(\pi_1(A), \pi_1(JB))J - \mathcal{J}_\alpha R(\pi_1(JA), \pi_1(B)) + R(\pi_1(JA), \pi_1(JB))J$$

The part of $N_\alpha(A^h, B^h)_J$ lying in $\mathcal{V}^*_J$ is the 1-form whose value at every vertical vector W is

$$\begin{aligned}(\mathcal{V}^* N_\alpha(A^h, B^h)_J)(W) = &-\tfrac{1}{2}[\pi_2(JA)(\pi_1(WB)) + \pi_2(WB)(\pi_1(JA)) \\ &+ (-1)^\alpha \pi_2(B)(\pi_1((J\circ W)A)) + (-1)^\alpha \pi_2((J\circ W)A)(\pi_1(B))] \\ &+ \tfrac{1}{2}[\pi_2(JB)(\pi_1(WA)) + \pi_2(WA)(\pi_1(JB)) \\ &+ (-1)^\alpha \pi_2(A)(\pi_1((J\circ W)B)) + (-1)^\alpha \pi_2((J\circ W)B)(\pi_1(A))].\end{aligned}$$

The endomorphism W of $T_pM \oplus T^*_pM$ is skew-symmetric with respect to the natural neutral metric and anti-commutes with J, so $\langle JA, WB\rangle =< (J\circ W)A, B >$. This gives

$$\pi_2(JA)(\pi_1(WB)) + \pi_2(WB)(\pi_1(JA)) = \pi_2(B)(\pi_1((J\circ W)A)) + \pi_2((J\circ W)A)(\pi_1(B)).$$

It follows that

$$\begin{aligned}\mathcal{V}^* N_\alpha(A^h, B^h)_J = &-\tfrac{1}{2}[1 + (-1)^\alpha][\pi_2(JA)(\pi_1(WB)) + \pi_2(WB)(\pi_1(JA)) \\ &- \pi_2(JB)(\pi_1(WA)) - \pi_2(WA)(\pi_1(JB))].\end{aligned}$$

This proves (i).

To prove (ii) take a section a of $A(M)$ near the point p such that $a(p) = V$ and $\nabla a|_p = 0$. Let $\tilde{a}$ be the vertical vector field defined by (9). Then it follows from Lemmas 6 and 7 that

$$N_\alpha(A^h, V)_J = \tfrac{1}{2} N_\alpha(A^h, \tilde{a})_J = ((J\circ V)A + (-1)^\alpha (J\circ V)A)^h_J.$$

To prove (iii) let us take the vertical coframe $\{\beta_{2r-1}, \beta_{2r}\}$, $r = 1, \ldots, 2n^2 - n$, defined before the statement of Lemma 8. Set $\varphi = \sum_{s=1}^{4n^2-2n} \varphi_s\beta_s$, $\varphi_s \in \mathbb{R}$. Let $E_1, \ldots, E_{2n}$ be a basis of T_pM and $\xi_1, \ldots, \xi_{2n}$ its dual basis. Then, by Lemma 8, we have

$$\begin{aligned}N_\alpha(A^h, \varphi)_J = \sum_{s=1}^{4n^2-2n} \varphi_s N_\alpha(A^h, \beta_s)_J = \sum_{s=1}^{4n^2-2n}\sum_{k=1}^{2n} \varphi_s\{[\beta_s(R(\pi_1(A), E_k)J) \\ + (-1)^{\alpha+1}\beta_s(J\circ R(\pi_1(JA), E_k)J)](\xi_k)^h_J + [\beta_s(R(\pi_1(JA), E_k)J) \\ - (-1)^{\alpha+1}\beta_s(J\circ R(\pi_1(A), E_k)J)](J\xi_k)^h_J\}.\end{aligned} \tag{10}$$

Moreover, note that

$$\langle \xi_k, B\rangle = \tfrac{1}{2}\xi_k(\pi_1(B)) \quad \text{and} < J\xi_k, B >= - < \xi_k, JB >= -\tfrac{1}{2}\xi_k(\pi_1(JB)),$$

therefore

$$\sum_{k=1}^{2n} \langle \xi_k, B\rangle E_k = \tfrac{1}{2}\pi_1(B), \quad \text{and} \quad \sum_{k=1}^{2n} < J\xi_k, B > E_k = -\tfrac{1}{2}\pi_1(JB).$$

Now (iii) is an obvious consequence of (10) and formula (i).

Finally, identity (iv) follows from the fact that the generalized almost complex structure $\mathcal{J}_\alpha$ on every fibre of $\mathcal{G}$ is induced by a complex structure. □

4. The integrability condition

Theorem 1. *Let M be a $2n$-dimensional manifold and ∇ a torsion-free linear connection on M. Let $\mathcal{J}_\alpha = \mathcal{J}_\alpha^\nabla$, $\alpha = 1, 2$, be the generalized almost complex structures induced by ∇ on the twistor space $\mathcal{G}$ of M. Then*

(i) *If $n = 1$, the structure $\mathcal{J}_1$ is always integrable.*
(ii) *If $n \geq 2$, the structure $\mathcal{J}_1$ is integrable if and only if the connection ∇ is flat.*
(iii) *The structure $\mathcal{J}_2$ is never integrable.*

Proof.

(i) Let $J \in \mathcal{G}$ and let $\{Q_1, Q_2 = JQ_1, Q_3, Q_4 = JQ_3\}$ be an orthonormal basis of $T_pM \oplus T_p^*M$, $p = \pi(J)$. By Proposition 1, the Nijenhuis tensor N_1 of $\mathcal{J}_1$ vanishes at the point J if and only if $N_1(Q_1^h, Q_3^h) = 0$ (in view of the fact that $N_1(\mathcal{J}_1 E, F) = N_1(E, \mathcal{J}_1 F) = -\mathcal{J}_1 N(E, F)$ for $E, F \in T\mathcal{G}$).

Let $\pi_1(Q_i) = e_i$, $i = 1, \ldots, 4$. Then, according to Proposition 1,

$$N_1(Q_1^h, Q_3^h) = [-R(e_1, e_3)J + R(e_2, e_4)J] - J \circ [R(e_1, e_4)J + R(e_2, e_3)J].$$

Both summands in the above formula vanish since, by Lemma 2, $e_3 = \cos t\, e_1 + \sin t\, e_2$, $e_4 = -\sin t\, e_1 + \cos t\, e_2$ for some $t \in \mathbb{R}$.

(ii) Let $\dim M = 2n \geq 4$ and assume that the generalized almost complex structure $\mathcal{J}_1^\nabla$ is integrable. Then, by Proposition 1, for every $p \in M$, every (genuine) complex structure K on T_pM and every $X, Y \in T_pM$ we have

$$R(X, Y)K + K \circ R(X, KY)K + K \circ R(KX, Y)K - R(KX, KY)K = 0,$$

where R is the curvature tensor of the connection ∇. The latter identity, as it is well-known, is the integrability condition for the Atiyah-Hitchin-Singer almost complex structure on the twistor space of complex structures on the tangent spaces of M (see, e.g., [28, Theorem 1 or Theorem 3]). Then, according to the arguments of [28, pp. 42-43], there exists a bilinear form μ on $TM \times TM$ such that

$$R(X, Y)Z = \mu(X, Y)Z - \mu(Y, X)Z + \mu(X, Z)Y - \mu(Y, Z)X. \tag{11}$$

Now let $p \in M$ and let $\{E_1, \ldots, E_{2n}\}$ be an arbitrary basis of T_pM. Denote by $\{\eta_1, \ldots, \eta_{2n}\}$ its dual basis. Let J be the complex structure on $T_pM \oplus T_p^*M$ for which $JE_{2k-1} = \eta_{2k}$, $JE_{2k} = -\eta_{2k-1}$, $k = 1, \ldots, n$. This structure is compatible with the natural neutral metric of $T_pM \oplus T_p^*M$, so we get a generalized complex structure sending T_pM onto T_p^*M and vice versa (it is similar to the structure in Example 2, Section 2). If n is an even number, the structure J yields the canonical orientation of $T_pM \oplus T_p^*M$, hence $J \in \mathcal{G}$. So, suppose that n is even and let $X, Y \in T_pM$. Then, by Proposition 1 (i), we have $R(X, Y)J = 0$. The latter identity is equivalent to the following identities

$$R(X, Y)\eta_{2k} - JR(X, Y)E_{2k-1} = 0, \;\; R(X, Y)\eta_{2k-1} + JR(X, Y)E_{2k} = 0,$$

$k = 1, \ldots, n$. It follows that for every $Z \in T_pM$, we have

$$2[\mu(X, Y) - \mu(Y, X)]\eta_{2k}(Z) + \mu(X, Z)\eta_{2k}(Y) - \mu(Y, Z)\eta_{2k}(X)$$
$$+ \mu(X, E_{2k-1})(JY)(Z) - \mu(Y, E_{2k-1})(JX)(Z) = 0$$

and

$$2[\mu(X,Y)-\mu(Y,X)]\eta_{2k-1}(Z)+\mu(X,Z)\eta_{2k-1}(Y)-\mu(Y,Z)\eta_{2k-1}(X)$$
$$-\mu(X,E_{2k})(JY)(Z)+\mu(Y,E_{2k})(JX)(Z)=0.$$

Let $k\neq l$, $1\leq k,l\leq n$, be two indexes. Putting $X=E_{2k-1}$, $Y=E_{2k}$, $Z=E_{2l-1}$ into the above identities, we get $\mu(E_{2k-1},E_{2l-1})=0$ and $\mu(E_{2k},E_{2l-1})=0$; putting $X=E_{2k}$, $Y=E_{2k-1}$, $Z=E_{2l}$, we get $\mu(E_{2k-1},E_{2l})=0$ and $\mu(E_{2k},E_{2l})=0$. Moreover, setting $X=Z=E_{2l-1}$, $Y=E_{2k}$ and $X=Z=E_{2l}$, $Y=E_{2k}$ into the first of the above identities, we obtain $\mu(E_{2l-1},E_{2l-1})=0$ and $\mu(E_{2l},E_{2l})=0$. Now take the basis $E'_1=E_1$, $E'_2=E_3$, $E'_3=E_2$, $E'_4=E_4,\ldots,E'_{2n}=E_{2n}$. Then the identities $\mu(E'_1,E'_3)=\mu(E'_3,E'_1)=0$ give $\mu(E_1,E_2)=\mu(E_2,E_1)=0$. It follows that $\mu(E_{2k-1},E_{2k})=\mu(E_{2k},E_{2k-1})=0$. Therefore $\mu=0$, thus $R=0$ by (11).

Now assume that $n=2m+1$ is an odd number. Let $X,Y\in T_pM$ be two linearly independent tangent vectors. Let $\{E_1,\ldots,E_{2n}\}$ be an arbitrary basis of T_pM with $E_1=X$, $E_2=Y$. Denote by $\{\eta_1,\ldots,\eta_{2n}\}$ its dual basis. Let J be the complex structure on $T_pM\oplus T^*_pM$ for which $JE_{2k-1}=\eta_{2k}$, $JE_{2k}=-\eta_{2k-1}$, $k=1,\ldots,m$, and $JE_{4m+1}=E_{4m+2}$, $J\eta_{4m+1}=\eta_{4m+2}$. Then $J\in\mathcal{G}$ and the preceding arguments show that

$$R(X,Y)E_i=0 \quad \text{for} \quad i=1,\ldots,4m.$$

Applying the latter identity for the basis $\{E_1,E_2,E_{4m+1},E_{4m+2},E_3,\ldots,E_{4m}\}$, we see that $R(X,Y)E_{4m+1}=R(X,Y)E_{4m+2}=0$. It follows that $R=0$.

(iii) Let $J\in\mathcal{J}$ and let $\{Q_1,Q_2=JQ_1,\ldots,Q_{4n-1},Q_{4n}=JQ_{4n-1}\}$ be an orthonormal basis of $T_pM\oplus T^*_pM$, $p=\pi(J)$. Define endomorphisms S_{ij} of $T_pM\oplus T^*_pM$ by (4). Then $V=S_{13}-S_{24}$ is a vertical vector of $\mathcal{G}$ at the point J and it follows from Proposition 1 (*ii*) that $N_2(E^h_1,V)_J=2(Q_4)^h_J\neq 0$. □

Remark 1. Concerning Theorem 1 (ii), let us note that that every flat torsion-free connection on a manifold induces an affine structure on it, i.e. local coordinates whose transition functions are affine. If, in addition, the connection is complete, then the manifold is the quotient of an affine space by a group of affine transformations acting freely and properly discontinuously on it (see, for example, [33]).

Remark 2. The complex structure $\mathcal{K}$ on the fibres of $\mathcal{G}$ is Kählerian with respect to the metric $\langle a,b\rangle=-\frac{1}{2}\mathrm{Trace}(a\circ b)$. Let $\mathcal{S}$ be the generalized complex structure on the vertical spaces of $\mathcal{G}$ induced by the Kähler form of the Kähler structure $(\mathcal{J},<,>)$ (see Example 2, Section 2). Then, given a connection ∇ on M, we can define two new generalized almost complex structures $\mathcal{I}_\alpha$ on the twistor space $\mathcal{G}$ by setting $\mathcal{I}_\alpha=(-1)^{\alpha+1}\mathcal{S}$ on $\mathcal{V}\oplus\mathcal{V}^*$ and $\mathcal{I}_\alpha=\mathcal{J}^\nabla_1\ (=\mathcal{J}^\nabla_2)$ on $\mathcal{H}\oplus\mathcal{H}^*$, $\alpha=1,2$. It is easy to see that the structures $\mathcal{I}_\alpha$ are never integrable. Indeed, let us adopt the notations used in part (*iii*) of the proof of Theorem 1. Then Lemma 6 implies that the projection of the Nijenhuis tensor $N_{\mathcal{I}_\alpha}(E^h_1,V)_J$ onto $\mathcal{H}_J$ is equal to $(Q_4)^h_J$.

Remark 3. The structure $\mathcal{J}^\nabla_1$ on the bundle of generalized complex structures inducing the orientation opposite to the canonical is integrable if and only if the torsion-free connection ∇ is flat, whereas $\mathcal{J}^\nabla_2$ is never integrable.

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